A lone researcher claims to have cracked one of the most famous problems in mathematics. Is it too good to be true, asks **Eric Kvaalen**

AS ONE of the editors of the *Annals of Mathematics*, Peter Sarnak sees his fair share of mathematical proofs. Yet there is one unsolved problem for which proofs keep on turning up in his mailbox. These are from people claiming to have cracked a long-standing conundrum known as the Riemann hypothesis. "At any given moment we probably have 10 claimed proofs submitted," says Sarnak, a mathematician at Princeton University.

Perhaps that is not so surprising. First put forward in 1859 by German mathematician Bernhard Riemann, the hypothesis is one of mathematics's most beguiling problems. Its allure lies in the fact that it holds the key to the primes, those numbers that underpin so much of today's mathematics. The Clay Mathematics Institute in Cambridge, Massachusetts, has deemed the problem so important that it is offering a $1 million prize to anyone who proves the hypothesis is true.

Despite mathematicians' best efforts, however, no one has yet published a proof of Riemann's hypothesis in a peer-reviewed journal. That might be about to change.

One respected mathematician claims to have cracked the problem, and has posted the proof on his website for others to scrutinise. If Louis de Branges of Purdue University in West Lafayette, Indiana, succeeds in getting his proof published in a journal, mathematicians will finally be able to sleep easier at night. "There are probably thousands of theorems in the literature which start, 'assume Riemann, then...', followed by some spectacular conclusion," says Sarnak.

At first glance, prime numbers look simple enough. They are whole numbers such as 11 and 13 that are only divisible by themselves and 1. Because every other whole number can be built by multiplying primes together, they are the most important numbers. Yet they are also the most frustrating. There is no obvious pattern to the primes, which makes finding new ones a laborious task that involves sifting through sets of numbers and testing them one by one.

In the 19th century, a surprising connection emerged between a sort of geometrical landscape and the way the primes are scattered among the numbers. The shape of this landscape is described by what is known as the Riemann zeta function. It works a bit like a mathematical GPS: give it your longitude and latitude, and the zeta function will tell you your elevation.

Riemann's discovery came when he began plugging "complex" numbers into the zeta function. Like map coordinates, complex numbers are made from two components: a real part – an ordinary number – and a so-called imaginary part, which is a multiple of *i*, the square root of -1. The complex numbers that produce a zero result – the mathematical equivalent of sea level – when put into the Riemann zeta function are known as its roots, and it turns out that the complex-number roots all have a real part equal to ½.

At least, that characteristic was true for all the roots Riemann looked at. He went on to surmise his result must be true for all the roots of the zeta function, but he didn't actually verify this hypothesis. "Of course one would wish for a stricter proof here," Riemann wrote. "I have for the time being... put aside the search for this, as it appears unnecessary for the next objective of my investigation." As it happens, he never did go back and prove it.

Even so, his hypothesis has stood the test of time. Researchers have used computers to find billions of roots, and they all obey Riemann's hypothesis – so it is now almost unthinkable that it is wrong. Sarnak goes further and says the Riemann hypothesis simply has to be true. "Its analogues are some of the most striking [mathematical] results we know from the 20th century, and it just wouldn't be right if somehow God made this wrong."

That said, finding billions of roots is not the same as proving that every possible root obeys Riemann's hypothesis. After all, there is an infinite supply of numbers. Indeed, some mathematicians speculate that things could still go wrong for very large complex numbers that have not yet been explored.

Deeper insights are needed, and over the past decade several purported proofs have been made publicly available. Yet only a proof which is accepted by a reputable mathematics publication and remains unchallenged for two years will prompt the Clay institute to pay out its $1 million bounty. None of the claims made so far has passed such scrutiny.

So why might de Branges have succeeded where so many others have failed? "The thing going for de Branges is that he solved another great problem," says Sarnak. That puzzle was first posed by the German mathematician Ludwig Bieberbach in 1916. It concerns an infinite sum of powers, such as the square of a number added to its cube, and so on. Bieberbach claimed that you can avoid the answer blowing up to infinity provided the terms in the sum obey certain conditions. He proved his conjecture was true for the first few terms, but he failed to prove it was always true. The problem troubled mathematicians until, in 1984, de Branges announced he had solved the Bieberbach conjecture.

Unfortunately, de Branges's proof was greeted with scepticism. Mathematicians complained that his paper was incoherent. Worse, de Branges's reputation among his peers had become tarnished. Two decades earlier, after becoming a professor at Purdue University, de Branges co-authored a paper with a student which turned out to contain a flaw. Instead of establishing him as a great mathematician, the work stigmatised him as someone too quick to publish.

To convince others that his Bieberbach proof was correct, de Branges spent three months at the prestigious Steklov Mathematical Institute in St Petersburg, Russia. There he worked with a group of experts who divided up the theorems in his paper and tried to check each one. Eventually they agreed de Branges had indeed found the way, although many of the theorems needed revising (*Acta Mathematica*, vol 154, p 137).

For nearly 25 years, de Branges has plugged away at the Riemann hypothesis. From time to time, he has posted on his website work he considers approaches a proof. Now he has gone one further and posted what he believes to be a proof of the Riemann hypothesis.

Key to this paper is the idea that there exists a deeper theory behind the zeta function. The type that Riemann studied involves positive whole numbers, but there are also other forms of zeta function that sum over mathematical sets. De Branges says he has found an underlying connection between them and this, he now claims, proves the Riemann hypothesis.

Other researchers are not so sure. The sins of de Branges's youth have not been forgiven, and it does not help that the man is a loner. Getting to grips with his paper requires understanding mathematics that he has developed more or less alone. Yashowanto Ghosh at Aquinas College in Grand Rapids, Michigan, is one of the few mathematicians to have studied de Branges's theory in depth. He says he is "fairly certain" that de Branges's approach is valid. "It differs from a valid proof only in that it lacks some technical details," says Ghosh.

Certainly de Branges knows he has his work cut out convincing his critics. The papers he has previously posted on his website tend to be overlong and seem repetitive, with almost no explanation of what he is trying to do or where he is going. Often he does not even bother to divide documents into sections. De Branges says he is making his Riemann hypothesis paper more readable by dividing it into sensible chunks and emphasising the key steps. Perhaps this is one proof that Sarnak won't mind receiving after all. ●

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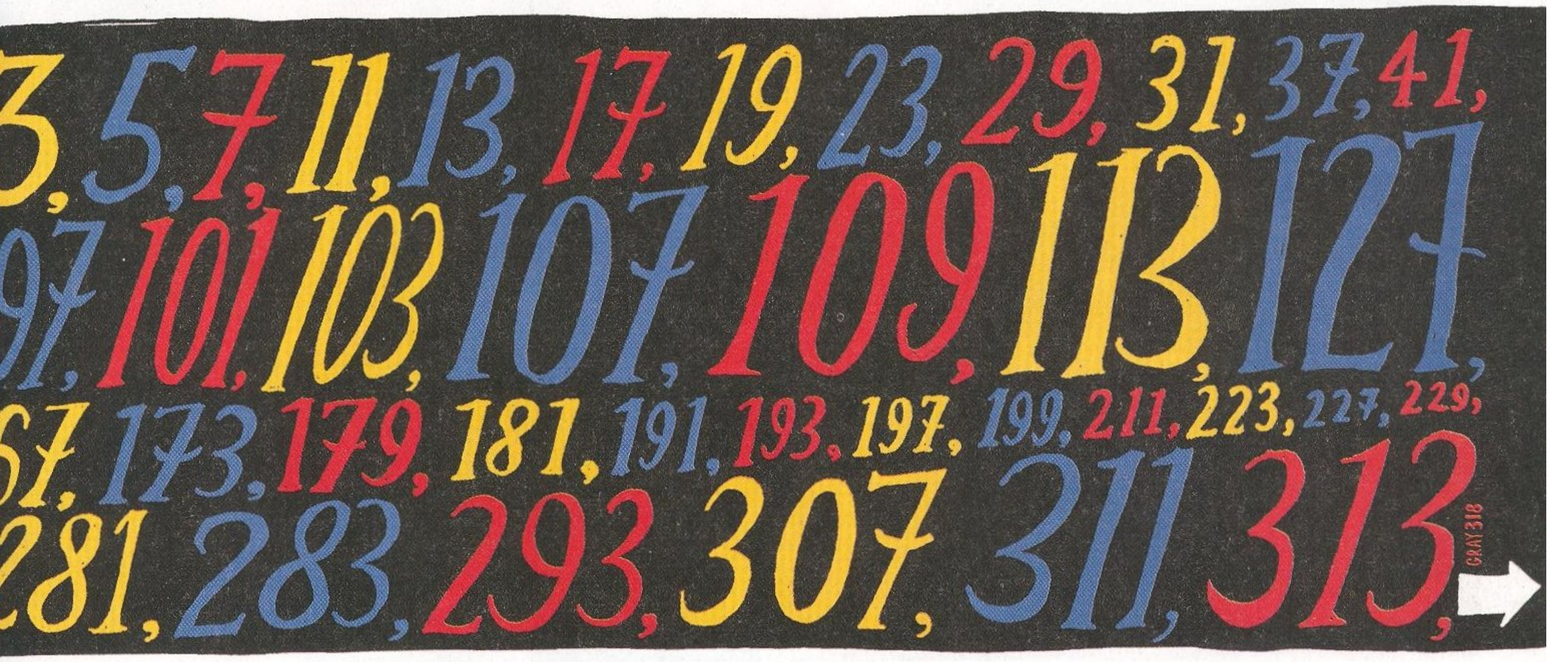
Further reading: "Riemann zeta functions" by Louis de Branges www.math.purdue.edu/~branges/riemann-hecke.pdf



Zeta and the primes

Though it was Bernhard Riemann who found that the zeta function "tames" the primes, the connection between them was discovered by Leonhard Euler, who showed that the zeta function (below) can be written as a product of infinitely many terms, each based on a prime number. Plugging any negative even number into the zeta function makes it go to zero, and the same happens when certain complex numbers, which have real and imaginary parts, are fed into it. Riemann noticed something remarkable about this set of complex numbers: when he plotted their real and imaginary parts on the complex plane, he found they all landed on the same line, with the real part equal to ½. Riemann's hypothesis is that every zero lies on this line. If this conjecture is true, it would allow us to pin down the distribution of prime numbers even more accurately.

$$\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots = \frac{2^n}{2^n-1}\,\frac{3^n}{3^n-1}\,\frac{5^n}{5^n-1}\,\frac{7^n}{7^n-1}\,\frac{11^n}{11^n-1}\cdots$$