
∇·E=0
∇×E= − 1/c ∂H/∂t
∇·H=0
20:03
iħ ∂/∂t ψ = Ĥψ
f̂(ξ) = ∫−∞∞ f(x)e−2πixξ dx

$$\nabla\times H=\frac{1}{c}\frac{\partial E}{\partial t}$$

# First among equals

Behind the scenes, equations rule our everyday lives. Mathematician Ian Stewart goes in search of the most influential

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

THE alarm rings. You glance at the clock. The time is 6.30 am. You haven't even got out of bed, and already at least six mathematical equations have influenced your life. The memory chip that stores the time in your clock couldn't have been devised without a key equation in quantum mechanics. Its time was set by a radio signal that we would never have dreamed of inventing were it not for James Clerk Maxwell's four equations of electromagnetism. And the signal itself travels according to what is known as the wave equation.

We are afloat on a hidden ocean of equations. They are at work in transport, the financial system, health and crime prevention and detection, communications, food, water, heating and lighting. Step into the shower and you benefit from equations used to regulate the water supply. Your breakfast cereal comes from crops that were bred with the help of statistical equations. Drive to work and your car's aerodynamic design is in part down to the Navier-Stokes equations that describe how air flows over and around it. Switching on its satnav involves quantum physics again, plus Newton's laws of motion and gravity, which helped launch the geopositioning satellites and set their orbits. It also uses random number generator equations for timing signals, trigonometric equations to compute location, and special and general relativity for precise tracking of the satellites' motion under the Earth's gravity.

DVARG/ALAMY

Without equations, most of our technology would never have been invented. Of course, important inventions such as fire and the wheel came about without any mathematical knowledge. Yet without equations we would be stuck in a medieval world.

Equations reach far beyond technology too. Without them, we would have no understanding of the physics that governs the tides, waves breaking on the beach, the ever-changing weather, the movements of the planets, the nuclear furnaces of the stars, the spirals of galaxies – the vastness of the universe and our place within it.

There are thousands of important equations. The seven I focus on here – the wave equation, Maxwell's four equations, the Fourier transform and Schrödinger's equation – illustrate how empirical observations have led to equations that we use both in science and in everyday life.

First, the wave equation. We live in a world of waves. Our ears detect waves of compression in the air as sound, and our eyes detect light waves. When an earthquake hits a town, the destruction is caused by seismic waves moving through the Earth.

Mathematicians and scientists could >

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hardly fail to think about waves, but their starting point came from the arts: how does a violin string create sound? The question goes back to the ancient Greek cult of the Pythagoreans, who found that if two strings of the same type and tension have lengths in a simple ratio, such as 2:1 or 3:2, they produce notes that, together, sound unusually harmonious. More complex ratios are discordant and unpleasant to the ear. It was Swiss mathematician Johann Bernoulli who began to make sense of these observations. In 1727 he modelled a violin string as a large number of closely spaced point masses, linked together by springs. He used Newton's laws to write down the system's equations of motion, and solved them. From the solutions, he concluded that the simplest shape for a vibrating string is a sine curve. There are other modes of vibration as well – sine curves in which more than one wave fits into the length of the string, known to musicians as harmonics.

## From waves to wireless

Almost 20 years later, Jean Le Rond d'Alembert followed a similar procedure, but he focused on simplifying the equations of motion rather than their solutions. What emerged was an elegant equation describing how the shape of the string changes over time. This is the wave equation, and it states that the acceleration of any small segment of the string is proportional to the tension acting on it. It implies that waves whose frequencies are not in simple ratios produce an unpleasant buzzing noise known as "beats". This is one reason why simple numerical ratios give notes that sound harmonious.

The wave equation can be modified to deal with more complex, messy phenomena, such as earthquakes. Sophisticated versions of the wave equation let seismologists detect what is happening hundreds of miles beneath our feet. They can map the Earth's tectonic plates as one slides beneath another, causing earthquakes and volcanoes. The biggest prize in this area would be a reliable way to predict earthquakes and volcanic eruptions, and many of the methods being explored are underpinned by the wave equation.

But the most influential insight from the wave equation emerged from the study of Maxwell's equations of electromagnetism. In 1820, most people lit their houses using candles and lanterns. If you wanted to send a message, you wrote a letter and put it on a horse-drawn carriage; for urgent messages, you omitted the carriage. Within 100 years, homes and streets had electric lighting, telegraphy meant messages could be transmitted across continents, and people even began to talk to each other by telephone. Radio communication had been demonstrated in laboratories, and one entrepreneur had set up a factory selling "wirelesses" to the public.

This social and technological revolution was triggered by the discoveries of two scientists. In about 1830, Michael Faraday established the basic physics of electromagnetism. Thirty years later, James Clerk Maxwell embarked on a quest to formulate a mathematical basis for Faraday's experiments and theories.

At the time, most physicists working on electricity and magnetism were looking for analogies with gravity, which they viewed as a force acting between bodies at a distance. Faraday had a different idea: to explain the series of experiments he conducted on electricity and magnetism, he postulated that both phenomena are fields which pervade space, change over time and can be detected by the forces they produce. Faraday posed his theories in terms of geometric structures, such as lines of magnetic force.

Maxwell reformulated these ideas by analogy with the mathematics of fluid flow. He reasoned that lines of force were analogous to the paths followed by the molecules of a fluid and that the strength of the electric or magnetic field was analogous to the velocity of the fluid. By 1864 Maxwell had written down four equations for the basic interactions between the electrical and magnetic fields. Two tell us that electricity and magnetism cannot leak away. The other two tell us that when a region of electric field spins in a small circle, it creates a magnetic field, and a spinning region of magnetic field creates an electric field.

But it was what Maxwell did next that is so astonishing. By performing a few simple manipulations on his equations, he succeeded in deriving the wave equation and deduced that light must be an electromagnetic wave. This alone was stupendous news, as no one had imagined such a fundamental link between light, electricity and magnetism. And there was more. Light comes in different colours, corresponding to different wavelengths. The wavelengths we see are restricted by the chemistry of the eye's light-detecting pigments. Maxwell's equations led to a dramatic prediction – that electromagnetic waves of all wavelengths should exist. Some, with much longer wavelengths than we can see, would transform the world: radio waves.

In 1887, Heinrich Hertz demonstrated radio

### THE ORIGIN OF EQUATIONS

**The ancient Babylonians and Greeks knew about equations, though they wrote them using words and pictures. For the past 500 years, mathematicians and scientists have used symbols, the crucial one being the equals sign. Unusually, we know who invented it, and why. It was Robert Recorde, who in 1557 wrote in his treatise *The Whetstone of Witte*: "To avoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or gemowe lines of one lengthe: bicause noe .2. thynges, can be moare equalle."**

FABIEN COURTTIARAT/REA/CONTRASTO/EYEVINE waves experimentally, but he failed to appreciate their most revolutionary application. If you could impress a signal on such a wave, you could talk to the world. Nikola Tesla, Guglielmo Marconi and others turned the dream into reality, and the whole panoply of modern communications, from radio and television to radar and microwave links for cellphones, followed naturally. And it all stemmed from four equations and a couple of short calculations. Maxwell's equations didn't just change the world. They opened up a new one.

Just as important as what Maxwell's equations do describe is what they don't. Although the equations revealed that light was a wave, physicists soon found that its behaviour was sometimes at odds with this view. Shine light on a metal and it creates electricity, a phenomenon called the photoelectric effect. It made sense only if light behaved like a particle. So was light a wave or a particle? Actually, a bit of both. Matter was made from quantum waves, and a tightly knit bunch of waves acted like a particle.

## Dead or alive

In 1927 Erwin Schrödinger wrote down an equation for quantum waves. It fitted experiments beautifully while painting a picture of a very strange world, in which fundamental particles like the electron are not well-defined objects, but probability clouds. An electron's spin is like a coin that can be half heads and half tails until it hits a table. Soon theorists were worrying about all manner of quantum weirdness, such as cats that are simultaneously dead and alive, and parallel universes in which Adolf Hitler won the second world war.

Quantum mechanics isn't confined to such philosophical enigmas. Almost all modern gadgets – computers, cellphones, games consoles, cars, refrigerators, ovens – contain memory chips based on the transistor, whose operation relies on the quantum mechanics of semiconductors. New uses for quantum mechanics arrive almost weekly. Quantum dots – tiny lumps of a semiconductor – can emit light of any colour and are used for biological imaging, where they replace traditional, often toxic, dyes. Engineers and physicists are trying to invent a quantum computer, one which can perform many different calculations in parallel, just like the cat that is both alive and dead.

Lasers are another application of quantum mechanics. We use them to read information from tiny pits or marks on CDs, DVDs and Blu-ray discs. Astronomers use lasers to measure the distance from the Earth to the moon. It might even be possible to launch space vehicles from Earth on the back of a powerful laser beam.

The final chapter in this story comes from an equation that helps us make sense of waves. It starts in 1807, when Joseph Fourier devised an equation for heat flow. He submitted a paper on it to the French Academy of Sciences, but it was rejected. In 1812, the academy made heat the topic of its annual prize. Fourier submitted a longer, revised paper – and won.

The most intriguing aspect of Fourier's prize-winning paper was not the equation, but how he solved it. A typical problem was to find how the temperature along a thin rod changes as time passes, given the initial temperature profile. Fourier could solve this equation with ease if the temperature varied like a sine wave along its length. So he represented a more complicated profile as a combination of sine curves with different wavelengths, solved the equation for each component sine curve, and added these solutions together. Fourier claimed that this method worked for any profile whatsoever, even a one where the temperature suddenly jumps in value. All you had to do was add up an infinite number of contributions from sine curves with more and more wiggles.

Even so, Fourier's new paper was criticised for not being rigorous enough, and once more the French academy refused to publish it. In 1822 Fourier ignored the objections and published his theory as a book. Two years later, he got himself appointed secretary of the

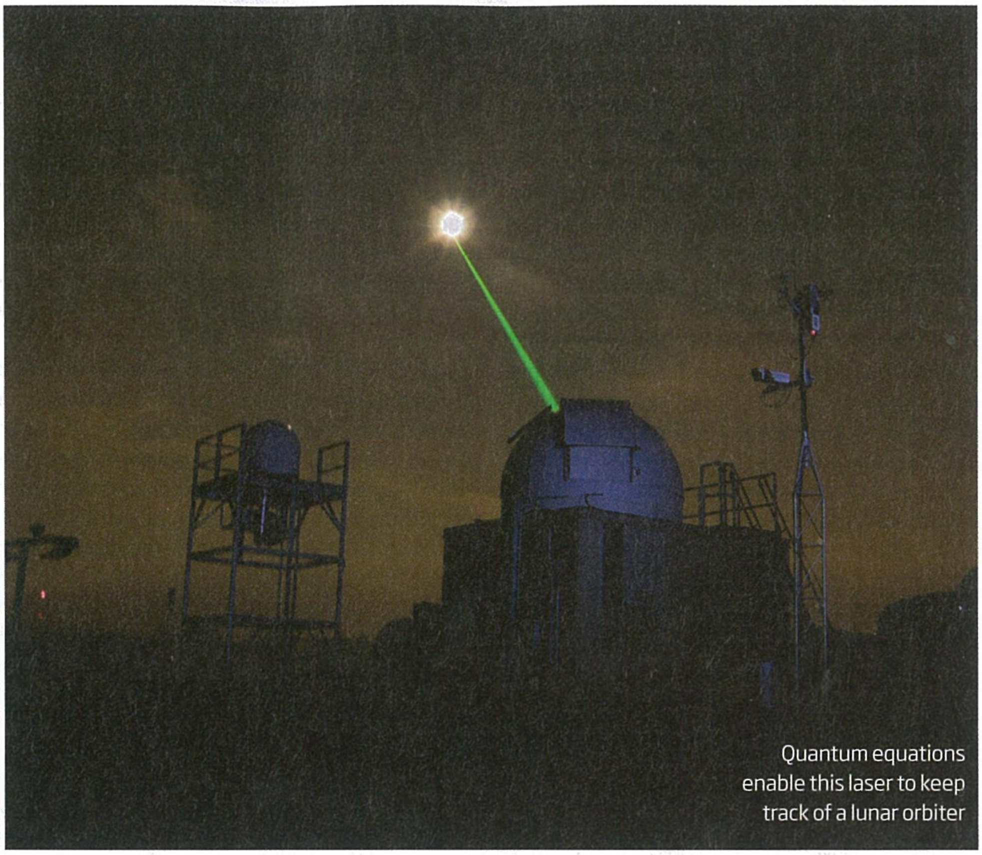
"Maxwell's equations led to the dramatic prediction that electromagnetic waves of all wavelengths should exist. Radio waves went on to transform the world"


The Pythagoreans figured out what makes strings sound harmonious

## THEOREMS AND THEORIES

Some equations present logical relations between mathematical quantities, and the task of mathematicians is to prove they are valid. Others provide information about an unknown quantity; here the task is to solve the equation and make the unknown known. Equations in pure mathematics are generally of the first kind: they reveal patterns and regularities in mathematics itself. Pythagoras's theorem, an equation expressed in the language of geometry, is an example. Given Euclid's basic geometric assumptions, Pythagoras's theorem is true.

Equations in applied mathematics and mathematical physics are usually of the second kind. They express properties of the universe that could, in principle, have been otherwise. For example, Newton's law of gravity tells us how to calculate the attractive force between two bodies. Solving the resulting equations tells us how planets orbit the sun or how to plot a trajectory for a space probe. But Newton's law isn't a mathematical theorem; the law of gravity might have been different. Indeed, it is different: Einstein's general relativity improves on Newton. And even that theory may not be the last word.



TOM ZAGWODZKI/GODDARD SPACE FLIGHT CENTER/NASA

Quantum equations enable this laser to keep track of a lunar orbiter

academy, thumbed his nose at his critics, and published his original paper in the academy's journal. However, the critics did have a point. Mathematicians were starting to realise that infinite series were dangerous beasts; they didn't always behave like nice, finite sums. Resolving these issues turned out to be distinctly difficult, but the final verdict was that Fourier's idea could be made rigorous by excluding highly irregular profiles. The result is the Fourier transform, an equation that treats a time-varying signal as the sum of a series of component sine curves and calculates their amplitudes and frequencies.

Today the Fourier transform affects our lives in myriad ways. For example, we can use it to analyse the vibrational signal produced by an earthquake and to calculate the frequencies at which the energy imparted by the shaking ground is greatest. A sensible step towards earthquake-proofing a building is to make sure that the building's preferred frequencies are different from the earthquake's.

Other applications include removing noise from old sound recordings, finding the structure of DNA using X-ray images, improving radio reception and preventing unwanted vibrations in cars. Plus there is one that most of us unwittingly take advantage of every time we take a digital photograph.

If you work out how much information is required to represent the colour and brightness of each pixel in a digital image, you will discover that a digital camera seems to cram into its memory card about 10 times as much data as the card can possibly hold. Cameras do this using JPEG data compression, which combines five different compression steps. One of them is a digital version of the Fourier transform, which works with a signal that changes not over time but across the image. The mathematics is virtually identical. The other four steps reduce the data even further, to about one-tenth of the original amount.

These are just seven of the many equations that we encounter every day, not realising they are there. But the impact of equations on history goes much further. A truly revolutionary equation can have a greater impact on human existence than all the kings and queens whose machinations fill our history books.

There is (or may be) one equation, above all, that physicists and cosmologists would dearly love to lay their hands on: a theory of everything that unifies quantum mechanics and relativity. The best known of the many candidates is the theory of superstrings. But for all we know, our equations for the physical world may just be oversimplified models that fail to capture the deep structure of reality. Even if nature obeys universal laws, they might not be expressible as equations.

Some scientists think that it is time we abandoned traditional equations altogether in favour of algorithms – more general recipes for calculating things that involve decision-making. But until that day dawns, if ever, our greatest insights into nature's laws will continue to take the form of equations, and we should learn to understand them and appreciate them. Equations have a track record. They really have changed the world and they will change it again. ■

"Today the Fourier transform affects our lives in myriad ways, from finding structures in DNA to compressing digital photographs"

**Ian Stewart** is a mathematician at the University of Warwick, UK. His latest book, *17 Equations That Changed the World*, is published by Profile